Intersection theorems on polytypes

van der Laan, G.; Talman, A.J.J.; Yang, Z.F.

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Intersection Theorems on Simplotopes *

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Gerard van der Laan, Dolf Talman, Zaifu Yang §

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†

Department of Econometrics and Tinbergen Institute, Free University, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands.

‡

Department

of
Econometrics
and
CentER,
Tilburg
University,
P.O.
Box
90153,
5000
LE
Tilburg,
The
Netherlands. §
Faculty
of
Business
Administration,
Yokohama
National
University,
79-
4
Tokiwadia,
Hodogaya-ku,
Yokohama
City
240-8501,
Japan.

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Abstract

Intersection
theorems
are
used
to
prove
the
existence
of
solutions
to
mathematical
programming
and
game
theoretic
problems.
The
known
intersection
theorems
on
the
unit
simplex
are
the
theorems
of
Knaster-Kuratowski-Mazurkiewicz
(KKM)
,
Scarf,

Shapley,
Freund,
and
Ichiishi.
Recently
the
intersection
result
of
KKM
has
been
generalized
by
Ichiishi
and
Idzik
to
closed
coverings
of a
compact
convex
polyhedron,
called a
polytope.
In
this
paper
we
formulate a
general
intersection
theorem
on
the
polytope.
To
do
so,
we
need
to
generalize
the
concept
of
balancedness
as
is
used
by
Shapley
and
by
Ichiishi.
The
theorem
implies
most
of
the
results
stated
above
as
special
cases.

First,
we
show
that
the
theorems
of
KKM,
Scarf,
Shapley,
Freund,
and
Ichiishi
on
the
unit
simplex
and
also
some
theorems
of
Ichiishi
and
Idzik
on a
polytope
all
satisfy
the
conditions
of
our
theorem
on
the
polytope.
Secondly,
the
general
theorem
allows
us
to
formulate
the
analogs
of
these
theorems
on
the
polytope.

Key
words:
intersection
theorem,
unit
simplex,
polytope,
closed
covering,
balancedness

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1
Introduction

Intersection theorems are used to prove the existence of solutions to mathematical programming problems and game theoretic problems. The probably most well-known intersection theorem is the Knaster-Kuratowski-Mazurkiewicz (KKM) lemma on the unit simplex. This lemma (see Knaster, Kuratowski and Mazurkiewicz [13]) states that n closed subsets covering the (n-1)-dimensional unit simplex Sn and satisfying some boundary condition have a nonempty intersection. A related theorem is due

to
Sperner
[20]
and
is
reformulated
by
Alexandroff
and
Pasynkoff
[1]
(see
also
[2,
4,
11,
18])
.
Further
generalizations
of
intersection
theorems
on
the
unit
simplex
can
be
found
in
Scarf
[17]
,
Shapley
[19]
,
Todd
[22]
,
Freund
[5]
,
Ichiishi
[8]
and
Joosten
and
Talman
[12]
.
Moreover,
generalizations
of
intersection
theorems
to
the
cube
or
simplotope
are
stated
in
Freund
[5]
,
van

der
Laan,
Talman
and
Van
der
Heyden
[15]
,
van
der
Laan
and
Talman
[16]
,
and
Talman
[21]
.
These
generalizations
can
be
used
to
prove
the
existence
of a
Nash
equilibrium
in a
noncooperative
N-person
game.

In
the
nineties
the
intersection
theory
has
been
generalized
by
Ichiishi
and
Idzik

[10]
to
closed
coverings
of a
compact,
convex
polyhedron,
called a
polytope.
In
this
paper
we
formulate a
general
intersection

theorem
on
the
polytope.
The
theorem
differs
from
the
results
mentioned
above
by
the
way
in
which
the
boundary
conditions
are
formulated.
In
all
these
intersection
theorems
the
boundary
condition
is
of
the
form
that
every
face
of
the
set
under
consideration
(simplex,
simplotope
or
polytope)
is
covered
by
the
union
of
some
specified
elements
of
the
family
of
sets
forming
the
covering.
In
our
theorem
we
give a

sufficient general condition to be satisfied at every point on the boundary. Our result implies most of the results stated above as special cases. In particular we show that the theorems of KKM, Alexandroff and Pasynkoff (AP in short) , Scarf, Shapley, Freund, and Ichiishi on the unit simplex and also some theorems of Ichiishi and Idzik on a polytope can be derived from our theorem as special cases.

To
our
knowledge,
there
is
no
result
which
has
unified
both
the
KKM
lemma
and
the
intersection
lemma
of
Alexandroff
and
Pasynkoff.
Our
theorem
also
allows
to
extend
some
theorems
on
the
unit
simplex
to
the
polytope.
In
particular,
we
formulate
the
analogs
of
the
theorems
of
KKM,
AP,
Shapley,
Freund,
and
Ichiishi
on
the
polytope.
One
of
these
results
is
closely
related
to a
result
given
in
Ichiishi

and
Idzik
[10]
.
This
paper
is
organized
as
follows.
Section 2
contains
the
mathematical
preliminaries.
Several
concepts
and
some
notation
are
introduced
concerning
the
polytope.
Also
the
concept
of
balancedness
of a
collection
of
vectors
is
given.
In
Section 3
we
formulate
and
prove
the
main
result.
In
Section 4
we
show
that
most
of
the
results
on
the
unit
simplex
follow
as
special
cases
from
our
main
theorem.
In
Section 5

we
derive
some
special
theorems
on
the
polytope.

3

2
Preliminaries

For a
convex
set B .
IRn ,
let
bnd(B)and
int(B)
denote
the
(relative)
boundary
and
the
(relative)
interior
of
B,
respectively.
For
any
positive
integer
n,
the
set
of
integers
{1,...,n}
is
denoted
by
In .
Consider
an
arbitrary
n-dimensional
polytope P
given
by

. .

P =
x.
IRn | a
i
.
x= a
i,
i.
I,

where

ai .
IRn
and a i .
IR,
i.
I,and I
is a
finite
set
of
at
least
n+
1
elements.
Throughout
the
paper
we
assume
that
none
of
the
constraints
defining P
is
redundant.
For T .
I,we
define
.

i..

F(T)
=
x. P |
ax=
ai
for
i.
T.

Observe
that
F(T)
, T .
I,may
be
empty.
When
F(T)
is
not
empty
we
call
F(T)aface
of
P.
When,
for
some i .
I, T =
{i},wecall
F(T)afacetof P

and we denote this facet by F_i. Observe that $F(\emptyset) = P$.

For x. bnd(P), the set I_x is defined as

.

i..

I_x = i. I | ax= a i ,

i.e. I_x is the set of indices for which the corresponding constraint is binding at x. Clearly, for any T . I x, x belongs to F(T). Furthermore, for nonempty T . I, let A(T) be the cone defined by

. .

A(T) =

x.
IRn |
x= . i a
i,.i =
0,
for
i. T

i.
T

and
define
A(Ø)
=
{0}
.
Moreover,
for
any T .
I,let
A*(T)bethe
polar
coneof
A(T)
defined
by

.

*..

A(T)
= y .
IRn |
yx= 0
for
all
x.
A(T) .

By
definition
we
have
that
A(T) n
A*(T)
=
{0},for
any T .
I.
Finally,
for
some
finite
set
J,
let a
collection
of
vectors
cj , j .
J,inIRn,
be
given.
For a
nonempty

set T .
J,we
define
..

.
..

j

C(T)
=
.
x.
IRn |
x=
μj
c,
μj
=
1
and
μj =
0, j .
T.
,

j.Tj.
T

i.e.
C(T)
is
the
convex
hull
of
the
vectors
cj , j .
T.
Definition
2.1

For
some
nonempty
subset T
of
J,
the
collection
of
vectors
{cj | j .
T}
is
balanced
when
the
origin
lies
in
the
relative
interior
of
C(T)

,
i.e.
if
the
system
of
linear
equations

.

*

j.
T
μj
cj
=
0
has a
positive
solution
μj , j .
T.

Without
confusion,
we
say
that
the
set T
is
balanced
if
the
corresponding
collection
of
vectors
is
balanced.

4

3
Intersection
theorem

In
this
section
we
present
our
main
result.
To
prove
the
main
result,
we
first
recall
the
following

lemma
which
can
be
seen
as a
version
of
Ky
Fan’s
coincidence
theorem.
This
theorem
has
been
used
to
prove
various
intersection
theorems
(see
Ichishi
[7]
and
Ichiishi
and
Idzik
[10])
.
For
the
proof
of
this
lemma
we
refer
to
Fan
[1972,
Theorem
5,
p.108])
.

Lemma
3.1
Let D
be a
nonempty,
compact,
convex
subset
of
IRn.Let .
and F
be
two
upper
semi-continuous
point-to-set
mappings
from D
to
IRn
such

that
for
any x . D
both
.(x)
and
F(x)
are
nonempty,
compact,
convex
subsets
of
IRn .
Suppose
that
for
each x . D

....

and
each c .
IRn
satisfying
cx
=max{cy | y .
D}
,
there
exist v .
.(x)
and w .
F(x)

....***

such
that
cv =
cw.
Then
there
is
at
least
one
point x . D
such
that
.(
x) n
F(
x)

=
Ø
.

For a
given
polytope P
and a
collection
of
vectors
{cj .

IRn | j .
J}
for a
finite
set
J,
supposewehavea
collection
of
sets
{Cj | j .
J}
being a
closed
covering
of
the
polytope P .
Given
this
collection,
for
any x . P
,we
define J x =
{
j . J | x .
Cj
}
.
In
the
next
theorem
we
give a
sufficient
condition
to
guarantee
that
there
is a
balanced
set
of
indices
in J
such
that
the
intersection
of
the
sets
labelled
by
these
indices
is
nonempty.

Main
Theorem

For a
finite
set

J,let
{cj .
IRn | j .
J}
be a
collection
of
vectors
and
let
{Cj | j .
J}
be a
collection
of
closed
sets
covering
the
full-dimensional
polytope P =
{
x .
IRn |
ai..
x = a
i, i .
I}
,
such
that
for
every x .
bnd(
P)
it
holds
that
C(
J
x) n
A*(
I
x)=Ø
.
Then
there
exists a
balanced
set T . J
for
which n
j.
T

* *
Cj =
Ø
.

Proof.
Define
the
two
point-to-set
mappings .
and F

from P
to
IRn
by

.(x)
=
{0}
,
F(x)
=
C(
J
x)
.

It
is
clear
that F
and .
have
nonempty,
convex
and
compact
values
and
are
upper
semi

....

continuous.
For x .
bnd(
P
)
,
take c .
IRn
such
that
cx
=max{cy | y . P
}
.
Then c .
A(
I
x)
.
By
the
boundary
condition,
there
exists z .
C(
J
x) n
A*(
I
x)
,
i.e.
there

exists

....

z .
F(x)
for
which
cz =
0=
cy
with y
=
0 .
.(x)
.
For x .
int(
P
)
,
we
have
that

....

cx
=max{cy | y . P }
if
and
only
if c
=
0
and
hence
it
follows
again
that
for
all

....

z .
F(x)wehavethat
cz =
0=
cy
with y
=
0 .
.(x)
.
Thus
all
conditions
of
Lemma

*

3.1
are
satisfied
and

there
exists a
point x . P
such
that
.(
x
*
) n
F(
x
*
)

= Ø
and
hence
*
*

**

0 .
C(
J x
)
.
Choosing a
subset T
of J x
such
that 0 .
int(C(
T
*))
,
we
have
that T
is

*

balanced
and n
j.
T

Cj =
Ø
.

5

In
the
next
section
the
Main
Theorem
will
be
applied
to

obtain
several
intersection
results
on
the
(n-
1)-dimensional
unit
simplex
in
the
n-dimensional
Euclidean
space.
To
do
so,
we
have
to
modify
the
main
theorem
for
allowing
to
apply
the
theorem
on
lower
dimensional
polytopes.
So,
for 1 =
l<n,
let
us
consider
the
l-dimensional
polytope
Pin
IRn
given
by

.

i
.
h..

P=
x.
IRn |
ax= a
i,
i.
Iand
dx= d
h,h. I
n-
l ,

h

where
d.
IRn
and d h .
IR,
h. I
n-
l
and
Iis a
finite
set
of
at
least
l+
1
elements.
Again
we
assume
that
none
of
the
constraints
defining
Pis
redundant.
To
adapt
the
main
theorem,
let

n-l

.

V=
{x.
IRn |
x= . h
dh,.h .
IR
for
h. I
n-l}

h=
1

and

*..
V=
{x.
IRn |
xy=0,for
all
y.
V}

as
the

subspace
orthogonal
to
V.
Furthermore,
for
nonempty
T.
I,let
A(T)bethe
cone
defined
by

. .

A(T)
=
V+
x.
IRn |
x= . i a
i,.i =
0,
for
i. T

i.
T

*

and
define
A(Ø)
=
V.Moreover,
for
any
T.
I,let
A(T)
be
again
the
polar
cone
of
A(T)
.
The
proof
of
the
next
corollary
goes
analogously
to
the
proof
of
the
main
theorem
by
using

....

thefact
that
for
thecasethat
x.
int(P)
wehavethatthe
vector
csatisfying
cx=
cy

*

for
all
y.
Plies
in
Vby
the
assumption
that
C(J) .
V.

Corollary
3.2
For a
finite
set
J,let
thecollectionofvectors
{cj .
IRn |
j.
J}
be
such
that
C(J) .
V*
and
let
{Cj |
j.
J}
be a
collection
of
closed
sets
covering a
polytope

*

Psuch
that
C(J) n
A(
I
x)
□
Ø
for

every
x.
bnd(P)
.
Then
there
exists a
balanced
set

X=

*

T* .
Jfor
which n
j.Tj

C=
Ø
.

4
Applications
to
the
unit
simplex

In
this
section
we
apply
Corollary
3.2
to
prove
several
well-known
intersection
results
on

n

nn

the
(n-1)-dimensional
unit
simplex
S=
{x.
IRn | x i
=1}.For
h. I
n,
Sdenotes

+
i=
1 h
nn

nn
the
facet
Sh =
{x.
S| x h
=0}
,
and
for
T. I
n,
S(T)
= n
h.
T
Sh .
Furthermore,
for

S
1

S. I
n,let
m.
IRn
be
defined
by
i.
S
ei,
where
|S|
denotes
the
number
of
elements

|S| i
Si

in
Sand
where
eis
the
ith
unit
vector
in
IRn .
Observe
that
m=
eif
S=
{i}.For
ease

I
n

of

notation
we
write
m=
m.
To
be
able
to
apply
Corollary
3.2,
observe
that
the
unit

n

simplex
Scan
be
rewritten
as
the
(n-
1)-dimensional
polytope
given
by

.

ni
.
1..

S=
x.
IRn |
ax= a
i,
i. I n
and
dx= d 1 ,

6

ii

where
a= m
-eand a i
=1/
n
for i
.
I n
and
where
d1 = m
and d 1
=1/n.Observe
that

*

n
ai
.
V
for
all i
.
I n
and
that F
(
T
)
=
S(
T
)for T
.
I n .
The
next
lemma
can
already
be
found
in
Alexandroff
and
Pasynkoff
[1]
and
is
equivalent
to a
lemma
of
Sperner
[20]
.

Theorem
4.1
AP
Lemma.

n

Let
{Cj
|
j
.
I
n}be a
collection
of
closed
sets
covering
the
unit
simplex
Ssatisfying
that
nh =
Ø

for
every h
.
I n
the
facet
Sh
is a
subset
of
C.
Then n
j.
I n
Cj
￿
.

Freund
[5]
obtained
the
following
result
by
relaxing
the
boundary
condition
of
the
AP
lemma.

Theorem
4.2
Freund
Lemma.

n

Let
{Cj
|
j
.
I
n}be a
collection
of
closed
sets
covering
the
unit
simplex
S.
Then
there

* n
*
*
k
exists
an x
.Ssuch
that

{
j
.In
|x>
0}.{
k
.In
|
x
.C}
.

j

Proof.
Let
C¯
j =
Cj
.Sjn
for
each j
.
I n .
Then
by
the
AP
lemma,
there
is
an x *
.
n
i.
I_n
C¯
i .

**

It
is
clear
that
x> 0
implies x
.Cj . .

j

The
AP
lemma
is a
special
case
of a
theorem
of
Scarf
[17]
which
is
implied
by
Corollary
3.2.

To
state
Scarf’s
theorem,
let B
be
an n
×k
matrix,
k>n,
satisfying
that

i

bi =
e, i
.In,
with
bj
the
jth
column
of
B, j
.Ik .

Theorem
4.3
Scarf
Lemma.

n

Let
{Cj
|
j
.
I
k}be a
collection
of
closed
sets
covering
the
unit
simplex
Ssatisfying
that

n

for
every h
.
I n
the
facet
Sis a
subset
of
Ch.Let c
.IRn
\{0}
be
given

and
assume

h +

that
the
set
of
solutions
{
y
.IRk
|By =
c}is
nonempty
and
bounded.
Then
there
exists

+

*

aset T
.
I k
such
that
the
system
of
equations
j.
T
µj
bj = c
has a
positive
solution
and

*

*

n
j.
T
Cj

=
Ø
.

Proof.
Take J = I k
and
for j
.
I
k,take
cj =
bj
-
. j

c,where . j =
nm..bj
.So . j
=
1
for

j
.
I n .
Without
loss
of
generality
we
may
assume
that c
.Sn .
Then
m..cj
=0, j
.
I
k,

*
n
and
hence
C(J)
.
V .
Let x
.bnd(S)and T =
Ix .
From
the
boundary
condition
we
have
that
C(
T)
.C(
J
x)
and
hence
the
boundary
condition
of
Corollary
3.2
is
satisfied
if
A*(
T
)nC(
T)
=
Ø
.
First

suppose
that c h
=
0
for
all h
.
T .
Then,
for
every
i,
j
.
T ,
it
holds
that
cj..ai
=(ej
-c)..(m-ei)
=
-ej..ei
=0,
so
that
cj
.A*(
T
)nC(
T
)for
any j
.
T
.Now

. c j
cj
suppose
that c h >
0for
some h
.
T .
Then
take y =
j.
T .
Clearly, y
.C(
T
)
.

S c

h.Th

..

Moreover,
for
every i
.
T ,
yai =
-yi

=0,
and
so y
.A*(
T
)
.
Hence,
the
conditions
of

*

*

Corollary
3.2
are
satisfied
and
there
is a
balanced
set T
.
I k
satisfying n
j.
T
Cj
=Ø.Bal

.

ancedness
of T *
implies
that
there
exist
.j * >
0for j
.
T *
satisfying
j.
T
.* j
=
1
such
that

*

7

.

*
*
*
*

*
ii

.cj
=0,
i.e.
.bj
-ac
=
0
with
a=
.: .
Since
b=
efor i
.
I
,

*
* *

j.Tj
j.Tj
j.Tjj
n

.
□□

*
*
i

we
have
that
c=
cbi,
from
which
it
follows
that
.bj -
acb=0.From

*

:
i.
I n i
.j.Tj
i.
I n i
the
boundedness
of
the
set
{
y
.IRk
|By =
c}it
follows
that 0

.{By
|
y
.IRk
\{0}}
.

+
+

*

Hence
a*
>0and
j.
T
µj
*bj =
c,
where
µj * = a j
>0for
all j
.T* .

*

*

The
following
lemma
is
the
lemma
of
Knaster,
Kuratowski
and
Mazurkiewicz
[13]
and
can
be
seen
as
the
dual
of
the
AP
lemma.

Theorem
4.4
KKM
Lemma.

n

Let
{Cj
|
j
.
I

n}be a
collection
of
closed
sets
covering
the
unit
simplex
Ssatisfying
that
for
every T
.
I
n,the
face
Sn(T)
is
contained
in .
j.
T
Cj .
Then n
j.
I n
Cj
.

=
Ø

The
KKM
lemma
is a
special
case
of
the
intersection
theorem
of
Shapley
[19]
.
To
formulate
Shapley’s
lemma,
we
need
the
concept
of
balancedness
of
sets.
To
distinguish
the
balancedness
of
sets
from
the
notion

of
balancedness
of
vectors
in
Definition
2.1,
we
speak
in
the
former
case
about
set-balancedness.

Definition
4.5

Let N
be
the
collection
of
all
nonempty
subsets
of
the
set
of
integers I n .
Then a
family

*

B=
{
B
1,...,Bk}of
k

elements
of N
is
set-balanced
if
there
exist
positive
numbers
.j ,

k
*

j
=1,...,k,
such
that
.mB j =
m.

j=
1 j

Theorem

4.6
Shapley
Lemma.

Sn

Let
{C| S
.N}
be a
collection
of
closed
sets
covering
the
unit
simplex
Ssatisfy

nS

ing
S(T)
.
.
S.
I n
\
T
Cfor
every T . I n .
Then
there
is a
set-balanced
family B =
kB j =
Ø

{
B
,...,B}of
elements
of N
for
which
nC
.

1k

j=
1

SS

Proof.
For
any
set S
.N, S
=
I
n,take
C=
m-m.
Then

C(J)
.V*.From
the

boundary
condition
we
have
that J x
contains
some
set S
.
I n
\T,where T = I x
.More-
S..

over,
for
any T
.
I
n,
caj
=0for
every j
.
T
and S
.
I n
\T.
Hence,
for
any x
on
the

n
*

boundary
of
Swe
have
that
C(
J)
nA(
I
x)
.
From
Corollary
3.2
it
follows
that
there

x=
Ø
exists a
family
B=
{B,...,Bk}satisfying

that
the
collection
of
vectors
cB j , j
=1,...,k,

1

kB j
S

is
balanced
and
nC=Ø
.
By
definition
of
the
vectors
c, S
.N,
the
family
Bis

j=
1

set-balanced
and
hence
the
theorem
holds. .

The
following
intersection
theorem
on
the
unit
simplex
is
due
to
Ichiishi
[8]
and
can
be
considered
as
the
dual
of
the
Shapley
Lemma.
The
proof
follows
analogously
by

taking

SS

C=
m-m, S
.N.

8

Theorem
4.7
Ichiishi
Lemma.

Sn

Let
{C|
S
.N}be a
collection
of
closed
sets
covering
the
unit
simplex
Ssatisfying

nS

that
for
every T
.In
it
holds
that
S(T)
..
T
.
S
C.
Then
there
is a
set-balanced
family

kB
j

B=
{
B
,...,Bk}of
elements
of N
for
which
nC=Ø
.

1

j=
1

The
next
lemma
is
due
to
Joosten
and
Talman
[12]
and
also
follows
from
Corollary

3.2
by
taking
cij =
ei
-ej ,
i, j
.
I n .
Theorem
4.8
Joosten
and
Talman
Lemma.

Let
{Cij | i . I
n, j . I
n}
be a
collection
of
closed
sets
covering
the
unit
simplex
Sn
satisfying
that
if x
.bnd(Sn)
,
then x
.Cij
for
some j
.
I n
with x j > 0
or x
.
.
j.
I
Cij

n

*

for
all i . I n
for
which x i
=0.
Then
there
exists a
nonempty
set
I. I n
and,
with
is

k =
|I*|
, a
permutation
s(I*)=(
s
1,...,sk)
of
the
elements
of
I*
,
such
that n
i.
I
C.

* i =
Ø
We
mention
that
the
intersection
theorems
of
AP,
KKM,
Shapley,
Freund,
and
Ichiishi
have
been
generalized
to
the
product
space
of N
unit
simplices
(see
[5,
14,
15,

16,
21])
.
We
notice
that
also
these
generalizations
can
be
derived
from
Corollary
3.2.

5
Intersection
theorems
on
the
polytope

In
this
section
the
Main
Theorem
is
applied
to
generalize
several
of
the
intersection
theo

i..

rems
on
the
unit
simplex
to
an
arbitrary
full
dimensional
polytope P =
{
x
.IRn|ax = a
i,
i .
I}
.
First
we
generalize
the
AP
Lemma
to
the
polytope.

To
prove
this
generalization,
we
state
the
following
lemma,
in
which
for T .
I,
A˜(T)
=
{
x .
IRn | x
=

i

. i
a, . i
=1, . i
=0,
i.T}
.

i.Ti.
T

Lemma
5.1
For
any T
.I,
A˜(T)
n-A*(T)

=
Ø
.

Proof.
If 0 .
A˜(T)
,
then
the
lemma
holds
because 0 .
A*(T)
.
Now,
suppose
that 0
.A˜(T)
.
Since
A˜(T)
.A(T)and
A(T)
nA*(T)
=

{0},it
follows
that
A˜(T)
nA*(T)
=
Ø
.

..
i
**

Hence,
for
every
x.A˜(T)
,
xa>
0
for
at
least
one
i.T.
Therefore,
a>
0
with
athe
optimal
value
of
the
primal
linear
programming
problem

..

(P)
min
a,
s.t.
x.A˜(T)and
a=xa
i,
for
all
i.T.
*

Now,
let
.be
the
optimal
value
of
the
dual
linear
programming
problem

..

(D)
max
.,
s.t.
-
y
.A˜(T)and .
=-ya
i,for
all
i.T.
*
*
*

Then
according
to
the
primal-dual
theory
we
have
that
.=
a>0.
Let y
be
any
solution

*..
i *
***

to
(D)
.
Then,
-ya=.>
0for
all i
.
T
and
hence y
.A(T)
.
Since
-
y
.A˜(T)
,

9

this
proves
the
lemma. .

Lemma
5.1
is
used
to

prove
the
following
generalization
of
the
AP
lemma
to
an
arbitrary
polytope P .

Theorem
5.2

Let a
polytope P
be
given
and
let
{Cj
|
j
.I}be a
collection
of
closed
sets
covering P

h

and
satisfying
that
for
every h
.
I
the
facet F h
is a
subset
of
C.
Then
there
exists a
set

*

*

T
.
I
such
that
{aj
|
j
.
T
*}is
balanced

and
nj.
T
Cj
□
=
Ø
.

Proof.
Take J = I
and
ch =
-ah , h
.I.
For a
boundary
point x
in
the
relative
interior

h

of F
(
T
)
,
we
have
that
-A˜(
T)
.C(
J
x)
since x
.Cfor
every h
.
T .
From
Lemma
5.1
it
follows
that
there
exists a y
.-A˜(
T)
satisfying y
.A*(
T
)
.
So, y
.C(
J
x)
nA*(
I
x)
,
since
-A˜(

T)
.C(
j
x)and I x = T .
Hence
the
boundary
condition
of
the
Main
Theorem
is

*
*

*

satisfied
and
there
exists a
balanced
set T
.
I m
for
which n
j.
T
Cj

=
Ø
.
Clearly,
if T
is

*

balanced,
then
also
{aj
|
j
.
T
}is
balanced. .

*

Theorem
5.2
says
that
under
the
boundary
condition
there
exists a
set
of
indices T

such

h
*

that
the
intersection
of
the
sets
C, h
.
T ,
is
not
empty
and 0
lies
in
the
interior
of

*

the
convex
hull
of
the
vectors
ah , h
.
T .
The
theorem
is
illustrated
in
Figure 1
for

*
*

n
=
2
and I = I
5.In
this
figure x
is
the
unique
intersection
point.
At x
it
holds
that

*
124
x
.CnCnCwhereas 0
lies

in
the
relative
interior
of
the
convex
hull
of
a1 ,
a2 ,
a4
and
hence
the
set
{1,
2,
4}is
balanced.
Notice
that
the
points
x1
and
x2
do
not
satisfy
the
requirements
because
at
x1
we
have
that 0
is
not
in
the
convex
hull
of
a2 ,
a3 ,
a4
and
at
x2
we
have
that 0
is
not
in
the
convex
hull
of
a1 ,
a4 ,
a5 .

In
Theorem
5.2

it
is
assumed
that
for
every h
.
I
the
facet F h
lies
in
Ch.The
next
corollary
follows
immediately
from
Theorem
5.2
by
taking
the
sets
covering P
equal
to
Cj
.Fj , j
.I.

Corollary
5.3
Let
{Cj
|
j
.I}be
an
arbitrary
collection
of
closed
sets
covering
the
polytope P .
Then
there
exist a
set T *
.
I
and
an x *
.
P ,
such
that
{aj
|
j
.
T
*}is

*
*

*

balanced
and
for
every j
.
T , x
.Cj
implies x
.
F j .

In
Corollary
5.3
we
allow
that
some
of
the
sets
Cj
are
empty.
In
particular,
suppose
that k
*

P
is
covered
by
just
one
set
Cfor
some k
.I.Now,
let x
be a
solution
to
the
linear
programming
problem

.
ki..

min
xa
subject
to
ax
=
b i
for
all i
.
I
(5.1)

10

Figure
1:
Illustration
of
Theorem
5.2; n =
2, I = I

5

i..*

and
let K
be
the
set
of
indices
given
by K =
{
i
.
I
|ax = b
i}
.
Then
it
follows
from

. *
**the
dual
of
(5.1)
that
there
exist a
set
J. K
and
positive
numbers
.i
for i .
Jsuch

*
ik *
**
that
i.
J
.i
a=
-aand
hence
with
.k = 1
we
have
that
T=

J.{k}
satisfies
the

*

**

conditions
with x
in
the
intersection
of
the
sets
Cj
.Fj , j
.T.

i..

In
case P
is
an
n-dimensional
simplex S
given
by S =
{
x
.IRn
|ax
=
b i
for i . I
}
,
we
have
that
according
to
Theorem
5.2
the
polytope
is
covered
by
n+ 1
closed
sets.

n+1

Since
for
any k =
n,
0
can
not
lie
in
the
relative

interior
of
the
convex
hull
of
any k
of
the
aj
's,
wemusthavethat
T* = I
and
hence
by
the
Main
Theorem
the
n+
1
sets

n+
1

have a
nonempty
intersection.
This
result
is
equivalent
to
the
AP
lemma
on
the
(unit)

*

simplex,
see
Theorem
4.1.
In
general,
if
in
Theorem
5.2
the
set
Tcontains
more
than
n+
1
elements,
then
the
set
T*
can
be

reduced
to a
set
T
.
of
n+ 1
indices,
such
that
the
vectors
aj , j .
T
,
are
affinely
independent.
Notice
that
we
do
not
require
that
in
Theorem
5.2
the
set
T*
must
exist
of
precisely
n+ 1
elements.
For
example,
let
the
polytope
be
given

n

by
the
n-dimensional
unit
cube
Kdefined
by

n

K=
{x.IRn
|
0
=
x i
=1for
i.
I
n}
.

nn
n

Let
us
denote
the
2nfacets
of
Kby F
+
i =
{x.K|
x i
=1}and F
-
i =
{x.K|
x i
=0}
,
for i
=1,...,n.
In
this
case I = I n
.-
I n
and
the
polytope
is
covered
by
2n
closed
sets,

+
i
-ih

denoted
by
Cand
Cfor
i=1,...,n,
such
that
Ccontains F h
for
every
h.
I n
.-
I n .
Under
this
boundary
condition
we
have
the
following
corollary
which
is

similar
to a
result

11

stated
in
[5]
.

Corollary
5.4
Let P
be
the
n-dimensional
unit
cube
Kn
and
let
{Ch
|
h
.
I n
.-
I
n}be a
collection
of
closed
sets
covering
Kn
such
that
for
every h
.
I n
.-
I n
the
facet F h
is a

+i

subset
of
Ch .
Then
there
is
an
index i
.
I n
such
that
CnC-
i
.

=
Ø

Theorem
5.2
can
be
seen
as
the
continuous
analog
of
Theorem 2
of
Freund
[6]
,
where
it
is
shown
that
in a
labelled
simplicial
subdivision
of
the
polytope P
with
label
set
I,there

h

exists a
set T
of
labels,
such
that 0
lies
in
the
convex
hull
of
the
set
{a, h
.T}and
there
is a
simplex s
such
that
the
set T
is
the
set
of
labels
of
the
vertices

of
s.
This
theorem
of
Freund
contains
as a
special
case a
result
of
van
der
Laan
and
Talman
[14]
,
implying
that
in
any
properly
labelled
simplicial
subdivision
of
an
n-dimensional
cube
Kn
with
label
set I n
.-
I
n,
there
exist
an
integer h
.
I n
and a
simplex s
having
two
vertices
with
labels h
and
-h.
This
result
is
the
combinatorial
analog
of
Corollary
5.4.
The
next
result
on
the
cube

is
due
to
Freund
[5]
.

Theorem
5.5
Freund
Theorem
on
the
Cube

Let P
be
the
n-dimensional
unit
cube
Kn
and
let
{Ch
|
h
.
I n
.-
I
n}
be a
collection
of
closed
sets
covering
Kn
such
that
for
every T
.
I n
.-
I
n,
with T
n{j,
-
j
}
=
Øfor
any j
.
I
n,
it
holds
that
{
x
.Kn
|
x j
=0,

j
.T,
xj
=1,
-
j
.T}..k.
T
Ck .
Then
there
is

+i

an
index i
.
I n
such
that
CnC-
i
=Ø
.

Proof.
We
have
that
a+
j =
ej
and
a-
j =
-ej
for
every j
.
I n
.Set J = I n
.-
I n
and
cj =
-aj
for j
.J.
Suppose
that x
is
an
arbitrary
interior
point
of a
proper
face
F(T)of
Kn .
Then
we
have
that

F(T)
=

{
z
.Kn
|
z j
=0,
j
.S,
zj
=1,
-
j
.
S }

with S = J
\{-
i | i . T
}
.
Observe
that
if i .
T,
then
-
i

. T
and
that
|T|
=
n.

*

Notice
that I x = T . S
and J x .
S.
It
is
easy
to
check
that
C(S) .
A(
I
x)
.
So
the

.
*boundary
condition
of
the
Main
Theorem
is
satisfied.
But
for
any
T. I n

.-
I
n,the

h

system
h.
T . h
a= 0
can
only
have a
positive
solution
if
for
some i
.
I n
both
+
i
and
-i

*

*

belong
to
T. .

In
the
next
theorem
we
generalize
the
KKM
lemma
on
the
unit
simplex
to
the

nn

polytope
P.
To
do
so,
we
first
observe
that
the
face
S(T)
of
the
unit
simplex

Sis
just

in\T

the
convex
hull
of
the
vertices
e, i
.T.
Hence
S(T)
can
be
written
as
.I n
with
.S
i

defined
as
the
convex
hull
of
the
vertices
e, i .
S.
Doing
so,
the
boundary
condition
of
the
KKM
lemma
can
be
rewritten
as
.S
.
.
j.
S
Cj
for
any S . I
n.This
gives
an

12

important
insight
in
the
difference

between
the
AP
lemma
and
the
KKM
lemma.
In
the

nh
h

AP
lemma,
each
facet
Sh
is
covered
by
Cand
so
the
number
of
sets
Cis
equal
to
the
number
of
facets.
In
generalizing
the
AP
lemma
to
the
polytope,
we
indeed
had
that
in
Theorem
5.2
the
number
of
sets
covering
the
polytope
equals
the
number
of
constraints.
However,
in
the
KKM
lemma
we

have
that
each
face,
being
the
convex
hull
of a
number
of
vertices,
is
covered
by
the
union
of
sets
corresponding
to
the
vertices
carrying
the
face.
So,
the
number
of
sets
equals
the
number
of
vertices.
For
n=
2
or
|I| =
n+ 1
the
number
of
vertices
equals
the
number
of
facets.
However,
for
n>2and
|I|
>n+
1,
this
is
in
general
not
the
case.
So,
generalizing
the
KKM

lemma
to
the
polytope,
the
number
of
sets
covering P
should
be
equal
to
the
number
of
vertices.
Therefore,
for a
given
polytope P =
{x.
IRn |
ai..x= a
i,
i.
I},let
tbe
the
number
of
vertices
and
let
{vj ,
j. I
t}
be
the
set
of
vertices
of
P.
Then
we
have
the
following
generalization
of
the
KKM
lemma,
in
which
the
boundary
condition
says
that
every
face
is
covered
by
the
sets

labelled
by
the
vertices
of
that
face.
The
proof
follows
again
straightforward
from
the
Main
Theorem
by
taking
J= I t
and
cj =
c-
vj ,
j.
J.
We
remark
that
this
result
can
also
be
derived
from
the
combinatorial
result
given
in
Theorem 4
of
Freund
[6]
.

Theorem
5.6

Let
{Cj | j . I
t}
be a
collection
of
closed
sets
covering
the
polytope P
with
vertices
vj ,

hh

j. I
t,

satisfying
that
every
face
F(T)
,
T.
I,
is
covered
by
.{C|
v.
F(T)}
.
Then
for
any
c. P
there
exists a
set
T* . I t
such
that
clies
in
the
convex
hull
of
the
vectors
vj ,
j.
T,and n
j.
T

*
*
Cj =
Ø
.

Theorem
5.6
holds
for
any
c.
P.If0 .
P,we
may
take
c=
0
and
obtain
the
result

*

that
there
is a

set
of
indices
Tfor
which
the
corresponding
sets
Cj
have a
nonempty
intersection
and
for
which
the
collection
of
corresponding
vertices
is
balanced.
Theorem

5.6
is
illustrated
in
the
Figure
2.
In
this
figure
we
have
that
the
point
clies
in
the
convex
135135
*

hull
of
v,
vand
v,
whereas
the
sets
C,
Cand
Cmeet
each
other
in
x.

It
should
be
noticed
that
Theorem

5.6 is a special case of Theorem 3.1 in Ichiishi and Idzik [10]. Ichiishi and Idzik also show that their Theorem 3.1 is a special case from the theorem below, stated as Theorem 3.4 in [10]. Below we show that this latter theorem is a special case of our Corollary 3.2. In the following, aff(S) denotes the affine hull of the set S.

Theorem 5.7 Ichiishi and Idzik

Theorem.

For
k>n,let
Bbe
an
n×
kmatrix
with
columns
bj ,
j. I
k,and
for
t,
n=
t=
k,let W
be
the
convex
hull
of
the
vectors
bj ,
j. I t
.Let
{Cj |
j. I
k}
be a
collection
of
closed
sets
covering
the
set
W.For
some
c.
W,assumethatthe
setofsolutions
{y.
IRk |
By=
c}

+

is
bounded.
Furthermore,
assume
that 0
.aff
(W)
,
bj .
aff
(W.{0}
)
for
all
j. I
k,and

13

Figure
2:
Illustration
of
Theorem
5.6; n =
2, I = I 5

that
for
every
proper
face F
of W
it
holds
that F
..{Cj
|bj
.aff
(F)+aff
{0,c}
, j
.
I
k}.

*

Then
there
exists a
set
T.
I k
such
that
the
system
of
equations
j.
T
µj
bj = c
has
a

*

*

positive
solution
and n
j.
T
Cj
.

=
Ø

Proof.
Since 0
.W,
without
loss
of
generality
we
may
assume
that
aff
(W)
=
{
x
.IRn
|

n

1

x j
=1}and
hence l = n
-1and
d=
m,
so
that V =
{
x
.

IRn
|
x =
.m, .
.IR}
.

j=
1
n

Now,
take J = I
k,
and
for j
.J,take
cj =
-bj + . j
c,where . j =
i=
1
bji .
Since c
.
W
we

....

have
that 0
.C(J)
.
Moreover,
mc
=1/
n
and
hence
for
all j
.
I k
we
have
that
mcj
=
0

*

and
therefore
cj
.
V
.So
C(J)
.
V
*.Now,
let x
be a
point
in
the
interior
of
some
face F
of
W.
Then, I x
.
I
is
the
set
of
elements
corresponding
to
its
binding
constraints,
i.e. F =
{
x
.
W |
ai..
x = a
i, i
.
I

x}
,
with
the
appropriate
vectors
ai
.IRn
and
numbers a i
.IR, i
.
I x .
According
to
the
boundary
condition
there
exists
an
element h
.
I k
such

hh

that
both
ch
.C(
J
x)and
b=
d+ d h
c,for
some
dh
.aff(F(
I
x)
)
and d h
.IR.
By
substi

hh
nh

tuting
bwe
now
obtain
that
c=
-dh
+(.h
-dh)c.
By
definition,
c=
0,
whereas

j=
1
j

□□□

h
nnn

both
dand c
are
in
W.Hence, 0 =
j=
1
chj =
j=
1
-dhj
+(
. h
-
d
h)
j=
1 c j =
-1+ . h
-
d h
and
so
ch =
c-dh.Moreover,
ai
. c
=
a i
for
all i
.
I
and
ai..dh = a i
for
all i
.
I
x.So,
for
all

*

i
.
I
x,
ai..ch =
ai..
c
-ai..dh
=0and
thus
ch
.A(
I
x)

.
Therefore,
ch
.A*(
I
x)
nC(
J
x)and

*

the
conditions
of
Corollary
3.2
are
satisfied.
Hence,
there
exists a
balanced
set
T.
I k
such

*
*
*

*

that n
j.
T
Cj

=
Ø
.
Balancedness
of
Timplies
that
there
exist
.j >
0for j
.Tsatisfying

. .

.*
cj
=0.
Hence,
.*bj =
a* c
with
a* =
.*
. .
Since
the
set

of
solutions

*
* *

j.Tj
j.Tj
j.
T
jj

**

{
y
.IRk
|By =
c}is
bounded,
we
must
have
that
a>
0.
Hence,
μbj =
c,
where

*

+
j.Tj
14

*

.

*
j
μ =
>
0
for
every j
.T* . .

ja *

Finally
we
consider
the
generalizations
of
the
Shapley
and
the
Ichiishi
lemma

on
the
unit
simplex
to
the
polytope.
The
first
case
is a
straightforward
generalization
of
Theorem
5.6
by
allowing
that
the
polytope P
is
covered
by a
collection
of
closed
subsets

SS

{C|
S
.L},where
Lis
the
collection
of
nonempty
subsets
of I t
.For S
.L,let
vbe

any
point
in
the
convex
hull
of
vertices
vj , j
.S.
Then
we
have
the
following
result
which
SS

follows
from
the
Main

Theorem
by
taking
C=
c-v, S
.L.

Theorem
5.8

Sh

Let
{C|
S
.L}be a
collection
of
closed
sets
covering
the
polytope P
with
vertices
v,

Sh

h.
I
t,
satisfying
that
every
face
F(T)
, T
.I,
is
covered
by
.{C|v.F(T)
for
all
h.
S}
.
Then,
for
any c
.P,
there
exists a
family
B=
{
B
,...,B}of k
elements
of
Lsuch

1kBk
Cj =
Ø

that c

lies
in
the
convex
hull
of
the
vectors
vj , j
=1,...,k,and
n
B

.

j=
1

Theorem
5.8
is
illustrated
in
Figure
3.
In
this
figure
the
point c
lies
in
the
convex
hull
of

1223
14512
23
145 *

v,
vand
v,
whereas
the
sets
C,
Cand
Cmeet
each
other
in
point x .
Observe

1

that
the
point
xdoes
not
satisfy
for
the
point c

chosen
in
the
figure,
but
becomes
an

1223
2

intersection
point
as
soon
as c
is
moved
to a
point
in
the
convex
hull
of
v,
vand
v.
To
generalize
the
lemma
of
Ichiishi
on
the
unit
simplex
to
the
polytope
P,let M

T

be
the
collection
of
subsets T
of I
for
which
F(T)is a
face
of
P,and
for T
.M,let
abe
any
point
in
A˜(T)
n-A*(T)
.
From
Lemma

5.1
we
know
that
the
latter
set
is
nonempty.

Th

Notice
that
a=
aif T =
{h}for
some
h.I.
Then
also
the
next
result
follows
from
the

SS

Main
Theorem
by
taking
c=
-a, S
.M.

Theorem
5.9

S

Let
{C| S
.M}
be a
collection
of
closed
sets
covering
the
polytope P
satisfying
that

S

every
face
F(T)
, T
.M,
is
covered
by
.{C| T .

S}
.
Then
there
exists a
balanced
Bk
Cj =
Ø

collection
of
vectors
aj , j
=1,...,k,
such
that
n
B

.

j=
1

SS
h

Observe
that
a.A˜(S)
and
hence
ais
the
convex
combination
of
the
vectors
a,
h.S.
This
implies
that
the
theorem
can
be
reformulated
by
stating
that
there
exists a
collection
B=
{
B
1,...,Bk}of
elements
of
Msuch
that
njk
=
1
CB j

=
Øand
{aj
|
j
.T*}is
balanced,

kT

where
T* =
.j=
1 B j .
This
immediately
proves
Theorem
5.2,
when
we
take
F(T)
=
Cin
case
|T|>1.

15

Figure
3:
Illustration
of
Theorem
5.8; n =
2, I = I

5

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